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ON Φ-MORI RINGS

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Abstract. A commutative ring R is said to be a ϕ-ring if its nilradical $Nil(R)$ is both prime and comparable with each principal ideal. The name is derived from the natural map ϕ from the total quotient ring $T(R)$ to R localized at $Nil(R)$. An ideal I that properly contains $Nil(R)$ is ϕ-divisorial if $(\phi(R) : (\phi(R) : \phi(I)) = \phi(I)$. A ring is a ϕ-Mori ring if it is a ϕ-ring that satisfies the ascending chain condition on ϕ-divisorial ideals. Many of the properties and characterizations of Mori domains can be extended to ϕ-Mori rings, but some cannot.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. For such a ring R, we let $T(R)$ denote the total quotient ring of R, $Z(R)$ denote the set of zero divisors of R and $Nil(R)$ denote the nilradical. We say that $Nil(R)$ is *divided* if it compares with each principal ideal of R (see [16] and [5]). Those ideals which are not contained in $Nil(R)$ are referred to as *nonnil ideals* (or are *nonnil*). If $Nil(R)$ is both divided and a prime ideal, we say R is a *ϕ-ring.* Note that each nonnil ideal of a ϕ-ring properly contains the nilradical. For convenience we let $\mathcal{H}$ denote the class of all ϕ-rings. The name is derived from the natural map $\phi : T(R) \to R_{Nil(R)}$ from the total quotient ring of R into R localized at $Nil(R)$. For $a, b \in R$ with b not a zero divisor, $\phi(a/b)$ is simply a/b viewed as an element of $R_{Nil(R)}$.

The elements in $R\backslash Z(R)$ are referred to as regular elements and an ideal I is said to be regular if it contains at least one regular element. For a nonzero ideal I, regular or not, we let $I^{-1} = \{x \in T(R) \mid xI \subset R\}$ and $I_v = (I^{-1})^{-1}$. The former

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is referred to as the *inverse* of I and the latter as the v of I. Both notations are dependent on knowing both the ideal and the ring in question at the time. For clarity, we occasionally use $(R : I)$ for the inverse. A nonzero ideal I of R is said to be *divisorial* if $I_v = I$.

Recall that a Mori domain is an integral domain that satisfies the ascending chain condition on divisorial ideals. Recently, the second-named author [28] generalized the concept of Mori domains to the context of commutative rings with zero divisors. According to [28] a ring R is called a *Mori ring* if it satisfies a.c.c on divisorial regular ideals. We are interested in extending the Mori property to ϕ-rings. Specifically we say that a nonnil ideal I is ϕ-divisorial if $\phi(I)$ is a divisorial ideal of $\phi(R)$, the image of R in $R_{Nil(R)}$. A ϕ-ring R is a *ϕ-Mori ring* if it satisfies a.c.c. on ϕ-divisorial ideals. Just as a field is trivially a Mori domain, if R is trivially a ϕ-ring in the sense that $Nil(R)$ is the maximal ideal of R, then R is considered to be a ϕ-Mori ring.

In Theorem 2.2 we show that a ϕ-ring R is ϕ-Mori if and only if $\phi(R)$ is a Mori ring. An alternate characterization is that R is ϕ-Mori if and only if $R/Nil(R)$ is a Mori domain (Theorem 2.5). As with Mori domains ([30, Théorème 1]) and Mori rings ([28, Theorem 2.22]), a ϕ-ring R is a ϕ-Mori ring if and only if for each nonnil ideal I, there is a finitely generated nonnil ideal $J \subset I$ such that $\phi(J)_v = \phi(I)_v$ (Theorem 2.14). For Mori domains, the statement is given in terms of nonzero ideals, and for Mori rings, it is given in terms of regular ideals. Other similarities to Mori domains include that if R is a ϕ-Mori ring, then R_P is ϕ-Mori for each nonnil prime P (Theorem 3.5). In Theorem 3.6 we show that if R is a ϕ-ring, then it is ϕ-Mori if and only if (i) R_M is ϕ-Mori for each maximal ϕ-divisorial ideal M, (ii) $\phi(R) = \cap\, \phi(R)_{\phi(M)}$ where the intersection is taken over the set of maximal ϕ-divisorial ideals, and (iii) each nonnil (ideal) element is contained in at most finitely many maximal ϕ-divisorial ideals. In the event there are no maximal ϕ-divisorial ideals, we assume the empty intersection is the ring $R_{Nil(R)}$. Note that the Mori ring in our Example 5.3 shows that a Mori ring need not be locally Mori.

One difference between Mori domains and ϕ-Mori rings is with regard to polynomial extensions. Of course there is no hope that the nilradical of $R[x]$ will be divided if R is not a domain, but in some cases the nilradical of $R(x)$ is a divided prime (see Examples 4.7, 5.4 and 5.7). Here $R(x)$ denotes the localization of $R[x]$ at the set of polynomials whose coefficients generate R as an ideal. It is known that if D is an integrally closed Mori domain, then $D[x]$ is a Mori domain [33, Théorème 3.5]. Since a localization of a Mori domain is a Mori domain, $D(x)$ is a Mori domain in this case. In Theorem 4.5 we show that if R is an integrally closed ϕ-Mori ring such that $Nil(R) = Z(R) \neq (0)$, then $R(x)$ is a ϕ-Mori ring if and only if each regular (equivalently, nonnil) ideal of R is invertible. Several examples show that this statement cannot be generalized to rings where the set of zero divisors properly contains the nilradical (see the aforementioned Examples 4.7, 5.4 and 5.7).

Throughout the paper we will use the technique of idealization of a module to construct examples. Recall that for an R-module B, the idealization of B over R is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a) + (s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A common notation for the idealized ring is $R(+)B$. See [20], [21] and [22] for basic properties of these rings.

A good reference for Mori domains is the recent survey article by V. Barucci [11].

2. Basic properties of ϕ-Mori Rings

We are concerned only with those rings for which the associated nilradical is a divided prime. For such a ring R, the kernel of the map $\phi : T(R) \longrightarrow R_{Nil(R)}$ is contained in $Nil(R)$ and it is a common ideal of R and $T(R)$. Since ϕ is a ring homomorphism, the ideals of R that contain $Ker(\phi)$ are in a one-to-one order preserving correspondence with the ideals of $\phi(R)$. Moreover, since the nilradical of $\phi(R)$ is simply the image of $\phi(Nil(R))$, there is a natural ring isomorphism between $R/Nil(R)$ and $\phi(R)/Nil(\phi(R))$. This isomorphism extends to a field isomorphism between the quotient fields of $R/Nil(R)$ and $R_{Nil(R)}/Nil(\phi(R)$ since $Nil(\phi(R)) = Z(\phi(R))$ is a divided prime of $\phi(R)$.

Lemma 2.1. *Let $R \in \mathcal{H}$ and let I, J be nonnil ideals of R. Then $I = J$ if and only if $\phi(I) = \phi(J)$.*

Proof. Since $Nil(R)$ is a divided prime of R and neither I nor J is contained in $Nil(R)$, both (properly) contain $Nil(R)$. Thus both contain the kernel of ϕ. The result follows from standard ring theory. □

A simple use of this lemma is the following characterization of ϕ-Mori rings in terms of Mori rings in the sense of [28].

Theorem 2.2. *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $\phi(R)$ is a Mori ring.*

Proof. Each regular ideal of $\phi(R)$ is the image of a unique nonnil ideal of R and $\phi(I)$ is a regular ideal of $\phi(R)$ for each nonnil ideal I of R. Moreover, by definition, if $J = \phi(I)$, then J is a divisorial ideal of $\phi(R)$ if and only if I is ϕ-divisorial. Thus a chain of ϕ-divisorial ideals of R stabilizes if and only if the corresponding chain of divisorial ideals of $\phi(R)$ stabilizes. It follows that R is ϕ-Mori ring if and only if $\phi(R)$ is a Mori ring. □

We recall the following lemma from [10, Lemma 1.1].

Lemma 2.3. *Let $R \in \mathcal{H}$. Then $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$.*

Our next lemma extends this result to $R_{Nil(R)}/Nil(\phi(R))$ and the quotient field of $R/Nil(R)$. Under the isomorphism we have that $(\phi(R) : \phi(I))/\phi(Nil(R))$ is isomorphic to $(R/Nil(R) : I/Nil(R))$ for each nonnil ideal I of R.

Lemma 2.4. *The following hold for each $R \in \mathcal{H}$.*

(a) *The map $\beta : \phi(R) \to R/Nil(R)$ given by $\beta(\phi(r)) = r + Nil(R)$ is a well-defined ring homomorphism whose kernel is $Nil(\phi(R))$.*

(b) *β extends to a map from $R_{Nil(R)}$ onto the quotient field of $R/Nil(R)$.*

(c) *Under the derived isomorphism $\widehat{\beta} : R_{Nil(R)}/Nil(\phi(R)) \to T(R/Nil(R))$, $(\phi(R) : \phi(I))/Nil(\phi(R))$ is isomorphic to $(R/Nil(R) : I/Nil(R))$ for each nonnil ideal I.*

(d) *For each nonnil ideal I of R, I is ϕ-divisorial if and only if $I/Nil(R)$ is a divisorial ideal of $R/Nil(R)$. Moreover, $\phi(I)$ is invertible if and only if $I/Nil(R)$ is invertible.*

(e) *For each nonempty set of nonnil primes $\mathcal{P}$ of R, $\phi(R) = \cap_{P \in \mathcal{P}} \phi(R)_{\phi(P)}$ if and only if $R/Nil(R) = \cap_{P \in \mathcal{P}} (R/Nil(R))_{P/Nil(R)}$.*

PROOF. For (a), all we need show is that β is well-defined. To this end let $r, s \in R$ be such that $\phi(r) = \phi(s)$. Thus $r - s \in Ker(\phi)$. As $Nil(R)$ contains $Ker(\phi)$, $r - s \in Nil(R)$ and therefore β is well-defined. Obviously, the kernel of β is the image of $Nil(R)$ in $\phi(R)$, but this is the nilradical of $\phi(R)$ since $Nil(\phi(R)) = \phi(Nil(R))$.

For (b), first recall that the nilradical of $\phi(R)$ is a common prime of $\phi(R)$ and $R_{Nil(R)}$. Moreover, it is the maximal ideal of $R_{Nil(R)}$ and is the entire set of zero divisors of both rings. That there is a natural isomorphism between $R_{Nil(R)}$ and the quotient field of $R/Nil(R)$ amounts to little more than the fact that, up to isomorphism, localizing at a prime commutes with moding out by the prime in question. What the extension of β to a map from $R_{Nil(R)}$ onto the quotient field of $R/Nil(R)$ amounts to is simply the composition of the canonical homomorphism from $R_{Nil(R)}$ onto $R_{Nil(R)}/Nil(\phi(R))$ and the isomorphism from $R_{Nil(R)}/Nil(\phi(R))$ onto $(R/Nil(R))_{Nil(R)/Nil(R)}$.

For (c), let I be a nonnil ideal of R. By way of the isomorphism in (b), it suffices to show that $\beta((\phi(R) : \phi(I)))/Nil(\phi(R))$ equals $(\phi(R)/Nil(\phi(R)) : \phi(I)/Nil(\phi(R)))$. We have that $(\phi(R)/Nil(\phi(R)) : \phi(I)/Nil(\phi(R)))$ contains $\beta((\phi(R) : \phi(I)))/Nil(\phi(R))$ since $\phi(I)$ properly contains $Nil(\phi(R))$, the latter a prime ideal of $\phi(R)$. That the two are equal is a consequence of the fact that $Nil(\phi(R))$ is divided and equal to the set of zero divisors of $R_{Nil(R)}$. This allows us to pull back each nonzero member of $(\phi(R)/Nil(\phi(R)) : \phi(I)/Nil(\phi(R)))$ to an element of $(\phi(R) : \phi(I))$.

Let s be a nonzero member of $(\phi(R)/Nil(\phi(R)) : \phi(I)/Nil(\phi(R)))$. Then there are nonnilpotent elements $a \in \phi(R)$ and $b \in \phi(I)$ such that s can be represented by multiplication by $(a + Nil(\phi(R))/(b + Nil(\phi(R))$. It suffices to show that $a/b \in (\phi(R) : \phi(I))$. Since $Nil(\phi(R)) = Z(R_{Nil(R)})$ is divided and b is not nilpotent, b is a regular element of $\phi(R)$ and $(a/b)Nil(\phi(R)) \subseteq Nil(\phi(R))$. For each $r \in \phi(I)$, we have $s(r + Nil(\phi(R)) = c + Nil(\phi(R))$ for some $c \in \phi(R)$. Thus $ar + Nil(\phi(R)) = bc + Nil(\phi(R))$ and from this we have $c - ar/b \in Nil(\phi(R))$ since b is a regular element of $\phi(R)$ and $Nil(\phi(R))$ is divided. It follows that $ar/b \in \phi(R)$.

The statements in (d) are a simple consequence of (c).

For (e), first suppose q is an element in $\cap\, \phi(R)_{\phi(P)} \backslash \phi(R)$. Then the ideal $J = (\phi(R) : (1, q))$ is a proper divisorial ideal of $\phi(R)$. Moreover for each $P \in \mathcal{P}$, J is not contained in $\phi(P)$. By definition, the inverse image of J in R is a proper ϕ-divisorial ideal of R, call it I. Then no $P \in \mathcal{P}$ contains I, and by (d), $I/Nil(R)$ is a proper divisorial ideal of $R/Nil(R)$. Making use of the map $\widehat{\beta}$, we see that the image of q in $T(R/Nil(R))$ is not in $R/Nil(R)$. On the other hand, q will be in $\cap\,(R/Nil(R))_{P/Nil(R)}$ since no prime from $\mathcal{P}$ contains I. Thus having $R/Nil(R) = \cap\,(R/Nil(R))_{P/Nil(R)}$ implies $\phi(R) = \cap\,\phi(R)_{\phi(P)}$. A similar proof establishes the converse. □

The following is a characterization of ϕ-Mori rings in terms of Mori domains.

Theorem 2.5. *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if $R/Nil(R)$ is a Mori domain.*

Proof. Suppose that R is a ϕ-Mori ring. Let $\{I_m/Nil(R)\}$ be an ascending chain of nonzero divisorial ideals of $R/Nil(R)$ where each I_m is a nonnil ideal of R. Hence $\{\phi(I_m)\}$ is an ascending chain of regular divisorial ideals of $\phi(R)$ by Lemma 2.4. Thus there exists an integer $n \geq 1$ such that $\phi(I_n) = \phi(I_m)$ for each $m \geq n$ and so we also have $I_n = I_m$ by Lemma 2.1 (for each $m \geq n$). It follows that $I_n/Nil(R) = I_m/Nil(R)$ as well.

Conversely, suppose that $R/Nil(R)$ is a Mori domain. Let $\{I_m\}$ be an ascending chain of nonnil ϕ-divisorial ideals of R. Thus $\{I_m/Nil(R)\}$ is an ascending chain of nonzero divisorial ideals of $R/Nil(R)$. Thus there exists an integer $n \geq 1$ such that $I_n/Nil(R) = I_m/Nil(R)$ for each $m \geq n$. As above, we have $I_n = I_m$ for each $m \geq n$. □

The following lemma makes it easy to show that each ϕ-Mori ring is also a Mori ring in the sense of [28].

Lemma 2.6. *Let $R \in \mathcal{H}$ and suppose that a nonnil ideal I of R is a divisorial ideal of R. Then $\phi(I)$ is a divisorial ideal of $\phi(R)$, i.e I is a ϕ-divisorial ideal of R.*

PROOF. Let $y \in \phi(I)_v$. Since $\phi(I)_v$ is contained in $\phi(R)$, $y = \phi(d)$ for some $d \in R$. We need to show that $y \in \phi(I)$. Now, let $x \in I^{-1}$. Since ϕ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, $\phi(x) \in (\phi(R) : \phi(I))$. Thus $y\phi(x) = \phi(d)\phi(x) = \phi(dx) = w \in \phi(R)$. Since the kernel of ϕ is a common ideal of R and $T(R)$, we must have $dx \in R$. As I is a divisorial ideal of R and x is an arbitrary element of I^{-1}, we have $d \in I$ and therefore $y \in \phi(I)$ as desired. □

Theorem 2.7. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring. Then R satisfies the a.c.c on nonnil divisorial ideals of R. In particular, R is a Mori ring.*

PROOF. Let $\{I_m\}$ be an ascending chain of nonnil divisorial ideals of R. Hence $\{\phi(I_m)\}$ is an ascending chain of regular divisorial ideals of $\phi(R)$ by Lemma 2.6. Thus there is an integer $n \geq 1$ such that $\phi(I_n) = \phi(I_m)$ for each $m \geq n$. Thus $I_n = I_m$ by Lemma 2.1. The "In particular" statement is now clear. □

The converse of Theorem 2.7 is not valid as can be seen by the following example.

Example 2.8. Let D be an integral domain with quotient field L which is not a Mori domain and set $R = D(+)(L/D)$, the idealization of L/D over D. Then $R \in \mathcal{H}$ is a Mori ring which is not a ϕ-Mori ring.

PROOF. First note that L/D is a divisible D-module; i.e., $d(L/D) = L/D$ for each nonzero $d \in D$. Thus for each nilpotent element $(0, b)$ and each nonnil element (d, f), there is an element $c \in L/D$ such that $(d, f)(0, c) = (0, dc) = (0, b)$. Since $Nil(R) = (0)(+)L/D$, it is a divided prime of R. Thus $R \in \mathcal{H}$. Since every nonunit of R is a zero divisor, we conclude that R is a Mori ring. On the other hand, since D is not a Mori domain and $R/Nil(R)$ is ring-isomorphic to D, $R/Nil(R)$ is not a Mori domain. Thus R is not a ϕ-Mori ring by Theorem 2.5. □

In the example section at the end of the paper, we will show how to construct a nontrivial Mori ring (i.e., where $R \neq T(R)$) in $\mathcal{H}$ which is not ϕ-Mori (Example 5.3).

Note that if $Nil(R) = Z(R)$, then $R_{Nil(R)}$ is simply the total quotient ring of R and $\phi(R) = R$. Thus we may state the following partial converse for Theorem 2.7.

Corollary 2.9. *Let $R \in \mathcal{H}$ such that $Nil(R) = Z(R)$. Then R is a ϕ-Mori ring if and only if R is a Mori ring.*

Recall from [10] that a ring $R \in \mathcal{H}$ is called a Nonnil-Noetherian ring if every nonnil ideal of R is finitely generated. Theorem 2.2 of [10] shows that a ring $R \in \mathcal{H}$ is a Nonnil-Noetherian ring if and only if $R/Nil(R)$ is a Noetherian domain. It is also the case that R is a Nonnil-Noetherian ring if and only if each regular ideal (equivalently, nonnil ideal) of $\phi(R)$ is finitely generated. At the time this paper was being written, the authors were aware of seven different types of ϕ-rings either in the literature or under study: (in alphabetical order) ϕ-Bezout, ϕ-Dedekind, ϕ-chained, ϕ-Krull, ϕ-Mori, ϕ-pseudo-valuation and ϕ-Prüfer (see [2], [3], [5], [7], and [8]). Letting "ϕ-BLANK" represent any one of these, it is known that a ring $R \in \mathcal{H}$ is a ϕ-BLANK ring if and only if $R/Nil(R)$ is a BLANK domain. Moreover, there are characterizations of each type in terms of the corresponding ring $\phi(R)$ as well. Thus it seems natural to change the name "Nonnil-Noetherian ring" to *ϕ-Noetherian ring*. We shall use this new name for the remainder of this paper.

Theorem 2.10. *Let $R \in \mathcal{H}$ be a ϕ-Noetherian ring. Then R is both a ϕ-Mori ring and a Mori ring.*

Given a Krull domain of the form $E = L + M$ where L is a field and M a maximal ideal of E, any subfield K of L gives rise to a Mori domain $D = K + M$. If L is not a finite algebraic extension of K, then D cannot be Noetherian (see [12, Section 4]). We make use of this in our next example to build a ϕ-Mori ring which is neither an integral domain nor a ϕ-Noetherian.

Example 2.11. Let K be the quotient field of the ring $D = \mathbb{Q} + X\mathbb{R}[[X]]$ and set $R = D(+)K$, the idealization of K over D. It is easy to see that $Nil(R) = \{0\}(+)K$ is a divided prime ideal of R. Hence $R \in \mathcal{H}$. Now since $R/Nil(R)$ is ring-isomorphic to D and D is a Mori domain but not a Noetherian domain, we conclude that R is a ϕ-Mori ring which is not a ϕ-Noetherian ring.

In light of Example 2.11, ϕ-Mori rings can be constructed as in the following example.

Example 2.12. Let D be a Mori domain with quotient field K and let L be an extension ring of K. Then $R = D(+)L$, the idealization of L over D, is in $\mathcal{H}$. Moreover, R is a ϕ-Mori ring since $R/Nil(R)$ is ring-isomorphic to D which is a Mori domain.

The following result is a generalization of [34, Theorem 1]. An analogous result holds for Mori rings when the chains under consideration are restricted to regular divisorial ideals whose intersection is regular [28, Theorem 2.22].

Theorem 2.13. *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if whenever $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq Nil(R)$, then $\{I_m\}$ is a finite set.*

PROOF. Suppose that R is a ϕ-Mori ring and $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq Nil(R)$. Hence $\{I_m/Nil(R)\}$ is a descending chain of (nonzero) divisorial ideals of $R/Nil(R)$ by Lemma 2.4 and $I_m/Nil(R) \neq \{0\}$ in $R/Nil(R)$. Since $R/Nil(R)$ is a Mori domain by Theorem 2.5, we conclude that $\{I_m/Nil(R)\}$ is a finite set by [34, Theorem 1]. Hence $\{I_m\}$ is a finite set. Conversely, suppose that whenever $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq Nil(R)$, then $\{I_m\}$ is a finite set. Since every nonzero ideal of $R/Nil(R)$ is of the form $I/Nil(R)$ for some nonnil ideal I of R and a nonnil ideal I of R is ϕ-divisorial if and only if $I/Nil(R)$ is a divisorial ideal of $R/Nil(R)$ by Lemma 2.4, we conclude that if $\{J_m\}$ is a descending chain of (nonzero) divisorial ideals of $R/Nil(R)$ such that $\cap J_m \neq \{0\}$, then $\{J_m\}$ is a finite set. Hence $R/Nil(R)$ is a Mori domain by [34, Theorem 1]. Thus R is a ϕ-Mori ring by Theorem 2.5. □

Mori domains can be characterized by the property that for each nonzero ideal I, there is a finitely generated ideal $J \subset I$ such that $(D : I) = (D : J)$ (equivalently, $I_v = J_v$) ([31, Theorem 1]). Our next result generalizes this result to ϕ-Mori rings.

Theorem 2.14. *Let $R \in \mathcal{H}$. Then R is a ϕ-Mori ring if and only if for any nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$, equivalently $\phi(J)_v = \phi(I)_v$.*

PROOF. Suppose that R is a ϕ-Mori ring and let I be a nonnil ideal of R. Since $R/Nil(R)$ is a Mori domain and $F = I/Nil(R)$ is a nonzero ideal of $R/Nil(R)$, there exists a nonzero finitely generated ideal L of $R/Nil(R)$ such that $L^{-1} = F^{-1}$. Since $L = J/Nil(R)$ for some nonnil finitely generated ideal J of R and $T(R/Nil(R)) = T(\phi(R))/Nil(\phi(R))$, we conclude that $\phi(J)^{-1} = \phi(I)^{-1}$.

Conversely, suppose that for each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$. Then one can see that for any nonzero ideal F of $R/Nil(R)$ there exists a nonzero finitely generated ideal L, $L \subset F$, such that $L^{-1} = F^{-1}$. Hence $R/Nil(R)$ is a Mori domain by [31, Theorem 1]. Thus, R is a ϕ-Mori ring by Theorem 2.5. □

In the following corollary we combine all of the different characterizations of ϕ-Mori rings stated in this section. In the next section we will give yet another characterization in terms of maximal ϕ-divisorial ideals (see Theorem 3.6).

Corollary 2.15. *Let $R \in \mathcal{H}$. The following statements are equivalent.*

(1) *R is a ϕ-Mori ring.*
(2) *$R/Nil(R)$ is a Mori domain.*
(3) *$\phi(R)/Nil(\phi(R))$ is a Mori domain.*
(4) *$\phi(R)$ is a Mori ring.*
(5) *If $\{I_m\}$ is a descending chain of nonnil ϕ-divisorial ideals of R such that $\cap I_m \neq Nil(R)$, then $\{I_m\}$ is a finite set.*
(6) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)^{-1} = \phi(I)^{-1}$.*
(7) *For each nonnil ideal I of R, there exists a nonnil finitely generated ideal J, $J \subset I$, such that $\phi(J)_v = \phi(I)_v$.*

3. Some properties of ideals of ϕ-Mori rings

The following result is a generalization of [34, Theorem 5].

Theorem 3.1. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and I be a nonzero ϕ-divisorial ideal of R. Then I contains a power of its radical.*

Proof. Set $D = R/Nil(R)$. Then D is a Mori domain by Theorem 2.5. Since $I/Nil(R)$ is a nonzero divisorial ideal of D by Lemma 2.4, we conclude that $I/Nil(R)$ contains a power of its radical by [34, Theorem 5]. Since $Nil(R)$ is divided, I contains a power of its radical. □

We recall a few definitions regarding special types of ideals in integral domains. For a nonzero ideal I of an integral domain D, I is said to be strong if $II^{-1} = I$, strongly divisorial if it is both strong and divisorial, and v-invertible if $(II^{-1})_v = D$. We will extend these concepts to the rings in $\mathcal{H}$.

Let I be a nonnil ideal of a ring $R \in \mathcal{H}$. We say that I is *strong* if $II^{-1} = I$, *ϕ-strong* if $\phi(I)\phi(I)^{-1} = \phi(I)$, *strongly divisorial* if it is both strong and divisorial, *strongly ϕ-divisorial* if it is both ϕ-strong and ϕ-divisorial, *v-invertible* if $(II^{-1})_v = R$ and *ϕ-v-invertible* if $(\phi(I)\phi(I)^{-1})_v = \phi(R)$. Obviously, I is ϕ-strong, strongly ϕ-divisorial or ϕ-v-invertible if and only if $\phi(I)$ is, respectively, strong, strongly divisorial or v-invertible.

The following lemma follows easily from Lemma 2.4 and the definitions above. We leave the proof to the reader.

Lemma 3.2. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and I be a nonnil ideal of R. Then the following hold.*

(a) *I is ϕ-strong if and only if $I/Nil(I)$ is a strong ideal of $R/Nil(R)$.*
(b) *I is strongly ϕ-divisorial if and only if $I/Nil(R)$ is a strongly divisorial ideal of $R/Nil(R)$.*
(c) *$(\phi(R) : \phi(I)) = \phi(R)$ if and only if $(R/Nil(R) : I/Nil(R)) = R/Nil(R)$.*
(d) *I is a ϕ-v-invertible ideal if and only if $I/Nil(R)$ is a v-invertible ideal of $R/Nil(R)$.*

In [31, Proposition 1], J. Querré proved that if P is a prime ideal of a Mori domain D, then P is divisorial when it is height one. In the same proposition, he incorrectly asserted that if the height of P is larger than one and P^{-1} properly contains D, then P is strongly divisorial. While it is true that such a prime must be strong, a (Noetherian) counterexample to the full statement can be found in [18]. What one can say is that P_v will be strongly divisorial (see [4]).

Theorem 3.3. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and P be a (nonnil) prime ideal of R. If $ht(P) = 1$, then P is ϕ-divisorial. If $ht(P) \geq 2$, then either $\phi(P)^{-1} = \phi(R)$ or $\phi(P)_v$ is strongly divisorial.*

Proof. Set $D = R/Nil(R)$ and let Q be a nonzero prime ideal of D. If Q is not strong, then QD_Q is principal (generated by t^{-1} where $t \in (D : Q)\backslash(Q : Q)$). Since D is a Mori domain (Theorem 2.5), D_Q is a quasilocal Mori domain whose maximal ideal is principal. Such a domain must be one-dimensional for otherwise the powers of the maximal ideal form an infinite descending chain of divisorial ideals with nonzero intersection. If $ht(Q) \geq 2$ and $(D : Q)$ properly contains D, then Q_v is strongly divisorial ([23, Proposition 2.2]).

Making use of Lemma 2.4 again, we have that if $ht(P) \geq 2$, then $\phi(P)$ is strong, and $\phi(P)_v$ is strongly divisorial whenever $(\phi(R) : \phi(P))$ properly contains $\phi(R)$.

On the other hand, if $ht(P) = 1$, then $ht(P/Nil(R)) = 1$. Moreover, $P/Nil(R)$ is a divisorial ideal of D by [31, Proposition 1]. Hence P is ϕ-divisorial by Lemma 2.4. □

For a ϕ-Mori ring $R \in \mathcal{H}$, let $D_m(R)$ denote the *maximal ϕ-divisorial ideals of R*; i.e., the set of nonnil ideals of R maximal with respect to being ϕ-divisorial. The following result generalizes [15, Theorem 2.3] and [12, Proposition 2.1].

Theorem 3.4. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring such that $Nil(R)$ is not the maximal ideal of R. Then the following hold.*

(a) *The set $\mathcal{D}_m(R)$ is nonempty. Moreover, $M \in \mathcal{D}_m(R)$ if and only if $M/Nil(R)$ is a maximal divisorial ideal of $R/Nil(R)$.*
(b) *Every ideal of $\mathcal{D}_m(R)$ is prime.*
(c) *Every nonnilpotent nonunit element of R is contained in a finite number of maximal ϕ-divisorial ideals.*

PROOF. Set $D = R/Nil(R)$. Then D is Mori domain by Theorem 2.5.

By Lemma 2.4, a nonnil ideal I of R is ϕ-divisorial if and only if $I/Nil(R)$ is a divisorial ideal of D. Thus we obviously have that I is a maximal ϕ-divisorial ideal of R if and only if $I/Nil(R)$ is a maximal divisorial ideal of D. A simple consequence of D satisfying the a.c.c. on divisorial ideals is that each (proper) divisorial ideal is contained in a maximal divisorial ideal. Moreover, each of its maximal divisorial ideals is prime [15, Theorem 2.3]. Hence $\mathcal{D}_m(R)$ is nonempty and each member of $\mathcal{D}_m(R)$ is a prime ideal of R.

Since every nonzero nonunit element of D is contained in a finite number of maximal divisorial ideals of D by [12, Proposition 2.1], it is easy to see that every nonnilpotent nonunit element of R is contained in a finite number of maximal ϕ-divisorial ideals of R. Thus (c) holds. □

As with a nonempty subset of R, a nonempty set of ideals $\mathcal{S}$ is *multiplicative* if (i) the zero ideal is not contained in $\mathcal{S}$, and (ii) for each I and J in $\mathcal{S}$, the product IJ is in $\mathcal{S}$. Such a set $\mathcal{S}$ is referred to as a multiplicative system of ideals and it gives rise to a generalized ring of quotients $R_{\mathcal{S}} = \{t \in T(R) \mid tI \subset R$ for some $I \in \mathcal{S}\}$. For each prime ideal P, $R_{(P)} = \{t \in T(R) \mid st \in R$ for some $s \in R \backslash P\} = R_{\mathcal{S}}$ where S is the set of ideals (including R) that are not contained in P. Note that in general a localization of a Mori ring need not be Mori (see Example 5.3 below). On the other hand if $\mathcal{S}$ is a multiplicative system of regular ideals, then $R_{\mathcal{S}}$ is a Mori ring whenever R is Mori ring ([28, Theorem 2.13]).

Theorem 3.5. *Let R be a ϕ-Mori ring. Then*

(a) *R_S is a ϕ-Mori ring for each multiplicative set S.*
(b) *R_P is a ϕ-Mori ring for each prime P.*
(c) *$R_{\mathcal{S}}$ is a ϕ-Mori ring for each multiplicative system of ideals $\mathcal{S}$.*
(d) *$R_{(P)}$ is a ϕ-Mori ring for each prime ideal P.*

PROOF. It suffices to prove (a) and (c).

Let S be a multiplicative set. No nilpotent element can be contained in S. Moreover, the nilradical of R_S is simply the localization of $Nil(R)$. Thus $Nil(R_S)$ is a divided prime of R_S. Also $(R_S)_{Nil(R_S)}$ is naturally isomorphic to $R_{Nil(R)}$. That R_S is ϕ-Mori now follows from Theorem 2.5.

Let $C = R\backslash Z(R)$. Then $R_C = T(R)$ is a ϕ-Mori ring. Note that if $\mathcal{S}$ is a multiplicative system of nonzero ideals that contains a subideal of $Nil(R)$, then $R_{\mathcal{S}} = T(R)$. Thus for (c) we may assume that each ideal of $\mathcal{S}$ is nonnil. In this case, each ideal in $\mathcal{S}$ properly contains $Nil(R)$ and we have that the set $\mathcal{T} = \{I/Nil(R) \mid I \in \mathcal{S}\}$ is a multiplicative system of ideals of $R/Nil(R)$. Since $R/Nil(R)$ is a Mori domain, $(R/Nil(R))_{\mathcal{T}}$ is a Mori domain [32, Théorème 2.2]. By Theorem 2.5, we also have that $T(R)/Nil(R)$ is also a Mori domain. We clearly have that $R_{\mathcal{S}}/Nil(R) = (R/Nil(R))_{\mathcal{T}} \cap T(R)/Nil(R)$, a finite intersection of Mori domains, and thus a Mori domain [32, Théorème 2.1]. That $R_{\mathcal{S}}$ is ϕ-Mori now follows from Theorem 2.5. □

One of the well-known characterizations of Mori domains is that an integral domain D is a Mori domain if and only if (i) D_M is a Mori domain for each maximal divisorial ideal M, (ii) $D = \cap D_M$ where the M range over the set of maximal divisorial ideals of D, and (iii) each nonzero element is contained in at most finitely many maximal divisorial ideals ([32, Théorème 2.1] and [34, Théorème I.2]). A similar statement holds for ϕ-Mori rings. Note that in condition (ii), if D has no maximal divisorial ideals, the intersection is assumed to be the quotient field of D. For the equivalence, that means that D is its own quotient field. The analogous statement is that if $\mathcal{D}_m$ is empty, then we have $R = T(R) = R_{Nil(R)}$ with $Nil(R)$ the maximal ideal.

Theorem 3.6. *Let $R \in \mathcal{H}$. Then the following are equivalent.*

(1) *R is a ϕ-Mori ring.*
(2) *(i) R_M is a ϕ-Mori ring for each maximal ϕ-divisorial M, (ii) $\phi(R) = \cap\, \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and (iii) each nonnil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals.*
(3) *(i) $R_{(M)}$ is a ϕ-Mori ring for each maximal ϕ-divisorial M, (ii) $\phi(R) = \cap\, \phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and (iii) each nonnil element (ideal) is contained in at most finitely many maximal ϕ-divisorial ideals.*

PROOF. First note that by Lemma 2.4, statement (ii) is equivalent to having $R/Nil(R) = \cap\,(R/Nil(R))_{M/Nil(R)}$ with the intersection again taken over the set of maximal ϕ-divisorial ideals of R. In both cases, we assume an intersection over an empty set of primes is the corresponding total quotient ring (quotient field for $R/Nil(R)$). That (1) implies both (2) and (3) now follows from [32, Théorème 2.1] and [34, Théorème I.2] together with Theorems 3.4 and 3.5 above. Theorem 3.5 also shows that (3) implies (2) since $R_M = (R_{(M)})_{R\setminus M}$. It remains to show that (2) implies (1).

Assume R_M is a ϕ-Mori ring for each maximal ϕ-divisorial ideal M, $\phi(R) = \cap\,\phi(R)_{\phi(M)}$ where the M range over the set of maximal ϕ-divisorial ideals, and that each nonnil element of R is contained in at most finitely many maximal ϕ-divisorial ideals. By Theorem 2.5, $R_M/Nil(R_M) \cong (R/Nil(R))_{M/Nil(R)}$ is a Mori domain. By Lemma 2.4, each maximal divisorial ideal of $R/Nil(R)$ is of the form $M/Nil(R)$ for some maximal ϕ-divisorial ideal M of R. Thus each nonzero element of $R/Nil(R)$ is contained in at most finitely many maximal divisorial ideals. Hence $R/Nil(R)$ is a Mori domain by [32, Théorème 2.1] and [34, Théorème I.2]. That R is a ϕ-Mori ring now follows from Theorem 2.5. □

Recall from [8] that a ring $R \in \mathcal{H}$ is called a *ϕ-chained ring* (*ϕ-CR*) if for every $x \in T(\phi(R))\setminus\phi(R) = R_{Nil(R)}\setminus\phi(R)$, we have $x^{-1} \in \phi(R)$; equivalently, if for every pair of elements $a, b \in R\setminus Nil(R)$, either $a \mid b$ or $b \mid a$ in R. If a ϕ-chained ring R has exactly one nonnil prime ideal and every nonnil ideal of R is principal, then we say that R is a discrete rank one ϕ-chained ring. We recall the following result from [2].

Lemma 3.7. ([2, Theorem 2.7]) *Let $R \in \mathcal{H}$. Then R is a ϕ-chained ring if and only if $R/Nil(R)$ is a valuation domain.*

If $Nil(R)$ a divided prime ideal of R, then one can easily prove the following result.

Lemma 3.8. *Let $R \in \mathcal{H}$ with Krull dimension different from zero. Then R is a discrete rank one ϕ-chained ring if and only if $R/Nil(R)$ is a discrete valuation domain.*

In [12], Barucci and S. Gabelli proved that if P is a maximal divisorial ideal of a Mori domain D, then the following three conditions are equivalent: (1) D_P is a discrete rank one valuation domain, (2) P is v-invertible, and (3) P is not strong [12, Theorem 2.5]. A similar result holds for ϕ-Mori rings. Lemma 3.2 is particularly useful in establishing this theorem.

Theorem 3.9. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and $P \in \mathcal{D}_m(R)$. Then the following statements are equivalent.*

1. *R_P is a discrete rank one ϕ-chained ring.*
2. *P is ϕ-v-invertible.*
3. *P is not ϕ-strong.*

PROOF. Set $D = R/Nil(R)$. Then D is a Mori domain by Theorem 2.5 and $P/Nil(R) \in \mathcal{D}_m(D)$.

To see that (1) implies (2), assume that R_P is a discrete ϕ-chained ring. From this it is easy to see that $D_{P/Nil(R)} \cong R_P/Nil(R_P)$ is a discrete valuation domain by Lemma 3.8. Hence $P/Nil(R)$ is v-invertible by [12, Theorem 2.5] and we have that P is ϕ-v-invertible by Lemma 3.2.

To see that (2) implies (3), first note that by Lemma 3.2, P is ϕ-v-invertible if and only if $P/Nil(R)$ is v-invertible, and P is ϕ-strong if and only if $P/Nil(R)$ is strong. As $P/Nil(R)$ is a divisorial ideal, it cannot be both strong and v-invertible. Thus (2) and (3) are equivalent.

If $P/Nil(R)$ is not strong, then $D_{P/Nil(R)} \cong R_P/Nil(R_P)$ is a discrete valuation domain [12, Theorem 2.3]. We will then have that R_P is a discrete ϕ-chained ring by Lemma 3.8. Hence (3) implies (1). □

The following result is a generalization of [4, Theorem 3.4].

Theorem 3.10. *Let $R \in \mathcal{H}$ be a ϕ-Mori ring and P be a nonnil prime ideal of R minimal over a nonnil principal ideal I of R. If P is finitely generated, then P has height one.*

PROOF. Set $D = R/Nil(R)$. Since D is a Mori domain and $P/Nil(R)$ is a minimal finitely generated prime ideal of D over the nonzero principal ideal $I/Nil(R)$ of D, we conclude that $ht(P/Nil(R)) = 1$ by [4, Theorem 3.4]. Hence it is clear that $ht(P) = 1$. □

4. THE SPECIAL CASE $Nil(R) = Z(R)$

While many of the results of this paper are true for the trivial case $Nil(R) = Z(R) = \{0\}$ (i.e., R is an integral domain), the main results in this section (Theorems 4.3 and 4.5 and their corollaries) are not. Moreover, they need not hold when we have that $Nil(R)$ is a nontrivial divided prime properly contained in $Z(R)$. A consequence of having $Nil(R) = Z(R)$ is that R will be a McCoy ring; i.e., each finitely generated ideal containing only zero divisors has a nonzero annihilator. Even more is true.

Theorem 4.1. *Let R be a ring for which $Nil(R) = Z(R)$. Then $Nil(R)$ is a prime ideal of R. Moreover, if $Nil(R)$ is a divided, then for each nonzero nilpotent m, $R/Ann(m)$ is such that $Nil(R/Ann(m)) = Z(R/Ann(m))$ is a divided prime of $R/Ann(m)$ and $R/Ann(m)$ is a McCoy ring.*

PROOF. The set of zero divisors of a ring is always a union of prime ideals. Thus if $Z(R) = Nil(R)$, $Nil(R)$ must be a prime ideal. Assume that it is divided and let m be a nonzero nilpotent element of R and let $\overline{R} = R/Ann(m)$. For each $r \in R$, let $\overline{r}$ denote the image of r in $\overline{R}$. Note that if n is a nilpotent element of R, $\overline{n}$ is nilpotent in $\overline{R}$. Since $Nil(R) = Z(R)$ is a prime ideal of R, $Ann(m) \subseteq Nil(R)$ and we have that $\overline{Nil(R)} = Nil(\overline{R})$ is a prime ideal of $\overline{R}$.

Let $r \in R$ be a regular element and let $s \in R$ be such that $\overline{r}\,\overline{s} = 0$. Then $rsm = 0$. Since r is regular, $sm = 0$. Thus $\overline{s} = 0$ and it follows that $Z(\overline{R}) = Nil(\overline{R})$, the nilradical of $\overline{R}$. For a nilpotent element $n \in Nil(R)$, $n/r \in Nil(R)$ since $Nil(R)$ is a divided prime. Thus $\overline{(n/r)} = \overline{n}/\overline{r}$ is in $\overline{R}$. It follows that $Nil(\overline{R}) = Nil(R)/Ann(m)$ is a divided prime of $\overline{R}$. □

The ring R constructed in Example 5.4 is in the set $\mathcal{H}$ but with $Nil(R) \neq Z(R)$. For each nonzero nilpotent m of this ring, $R/Ann(m)$ is a McCoy ring. However, it is rare that $R/Ann(m)$ is such that its nilradical is both a divided prime and equal to $Z(R/Ann(m))$. It is possible to have $Nil(Ann(m))$ equal to $Z(R/Ann(m))$ but not be divided, and it is possible to have $Nil(R/Ann(m))$ neither divided nor equal to $Z(R/Ann(m))$ (see the above mentioned Example 5.4).

Theorem 4.2. *Let R be an integrally closed ring for which $Nil(R) = Z(R)$. Then $Nil(R)$ is a divided prime of R, $R/Nil(R)$ is an integrally closed domain and $R/Ann(m)$ is integrally closed for each nonzero nilpotent m.*

PROOF. Since R is integrally closed and $Nil(R) = Z(R)$, each nonnil element of R is regular. Moreover, for $r \in R\backslash Nil(R)$ and $m \in Nil(R)$, m/r must be in R. Hence $Nil(R)$ is a divided prime of R. We also have that $Nil(R)$ is the maximal ideal of $T(R)$. Thus $T(R/Nil(R)) = T(R)/Nil(R)$ and since R is integrally closed in $T(R)$ and $Nil(R)$ is a common prime ideal of R and $T(R)$, $R/Nil(R)$ is an integrally closed domain. Let m be a nonzero nilpotent of R and let $\overline{R} = R/Ann(m)$. By the previous theorem, $Z(\overline{R}) = Nil(\overline{R}) = \overline{Nil(R)}$ is a divided prime of $\overline{R}$. Moreover, $Ann(m)$ is a common ideal of R and $T(R)$, and $T(\overline{R}) = T(R)/Ann(m)$. It follows that $\overline{R}$ is integrally closed in $T(\overline{R})$. □

In our next two theorems and the corresponding corollaries, we assume that $Nil(R)$ is not trivial. Borrowing notation from [25], we let NT denote the nilradical of $T(R[x])$. Recall that a ring R is said to be a Prüfer ring if each finitely generated regular ideal is invertible [22].

Theorem 4.3. *Let R be an integrally closed ring for which $Nil(R) = Z(R) \neq \{0\}$. Then the following are equivalent.*

(1) *$R(x)$ is integrally closed.*
(2) *$R(x)$ contains NT.*
(3) *$R/Nil(R)$ is a Prüfer domain.*
(4) *For each maximal ideal M, R_M is a ϕ-chained ring.*
(5) *R is a Prüfer ring.*

PROOF. Since R is integrally closed and $Nil(R) = Z(R)$, $Nil(R)$ is divided and each ideal that is not contained in $Nil(R)$ is regular and contains $Nil(R)$. Moreover, R is a McCoy ring.

If $R(x)$ is integrally closed, then it must contain the nilradical of $T(R[x])$. Thus (1) implies (2).

If R is also a Prüfer ring, then $R(x)$ is integrally closed [22, Theorem 16.8] and we have that (5) implies (1).

Another simple implication is that (4) implies (5). If I is a finitely generated regular ideal of R, then the same can be said for each IR_M. If R_M is ϕ-chained, then IR_M is principal. From this it follows that each finitely generated regular ideal of R is invertible. Thus R is a Prüfer ring.

By Theorems 4.1 and 4.2, we have that $R/Ann(m)$ is an integrally closed McCoy ring for each nonzero nilpotent m and that $R/Nil(R)$ is an integrally closed domain. We also have that $Nil(R)$ contains $Ann(m)$ and that $Nil(R/Ann(m)) = Nil(R)/Ann(m) = Z(R/Ann(m))$ is a divided prime of $R/Ann(m)$. Moreover, the following are equivalent for an ideal I that is not contained in $Nil(R)$, (i) I is finitely generated, (ii) $I/Nil(R)$ is finitely generated and (iii) $I/Ann(m)$ is finitely generated. All three of the ideals I, $I/Nil(R)$ and $I/Ann(m)$ are regular.

To see that (2) implies (3), assume $R(x)$ contains NT and let I be an ideal whose image in $R/Nil(R)$ is a nonzero finitely generated ideal. In addition, let m be a nonzero nilpotent. By the above, both I and $I/Ann(m)$ are finitely generated regular ideals. Since $R/Ann(m)$ is an integrally closed McCoy ring, $I/Ann(m)$ must be an invertible ideal of $R/Ann(m)$ [25, Theorem 8]. It follows that $R/Ann(m)$ is a Prüfer ring whose nilradical is a divided prime. Thus $R/Nil(R)$ is a Prüfer domain [2, Theorem 2.6].

To complete the proof we show that (3) implies (4). To this end, assume $R/Nil(R)$ is a Prüfer domain and let M be a maximal ideal of R. Localizing $R/Nil(R)$ at a maximal $M/Nil(R)$ will yield a valuation domain. As $(R/Nil(R))_{M/Nil(R)}$ is naturally isomorphic to $R_M/Nil(R_M)$, the result follows from [2, Theorem 2.7] (see Lemma 3.7 above). □

Corollary 4.4. *Let R be a ring for which $Nil(R) = Z(R) \neq (0)$. If $R(x)$ contains NT, then $Nil(R)$ is a divided prime of R and the integral closure of R is a Prüfer ring.*

Proof. Let R' be the integral closure of R. Since NT contains $Nil(T(R))$ and $R(x) \cap T(R) = R$, $Nil(R) = Nil(T(R))$. Since $Nil(R) = Z(R)$, we have that $Nil(R)$ is a divided prime of R. Obviously, we must have $Nil(R) = Nil(R') = Z(R')$ and $NT \subseteq R'(x)$. Thus R' satisfies the five equivalent statements of the previous theorem. In particular, R' must be a Prüfer ring. □

A ring R is said to be a *ϕ-Dedekind ring* if $Nil(R)$ is a divided prime of R and for each nonnil ideal I of R, $\phi(I)$ is an invertible ideal of $\phi(R)$ [3]. It is known that if $Nil(R)$ is a divided prime, then R is a ϕ-Dedekind ring if and only if $R/Nil(R)$ is a Dedekind domain [3, Theorem 2.5]. Our next result provides several other characterizations for the special case that $Nil(R) = Z(R) \neq \{0\}$. Later we will construct an example of an integrally closed ϕ-Noetherian ring R with $\{0\} \neq Nil(R) \neq Z(R)$ such that $R(x)$ is ϕ-Noetherian and $R/Nil(R)$ is an integrally closed Noetherian domain that is not Dedekind (Example 5.4). We also give an example of an integrally closed ϕ-Mori ring R with $\{0\} \neq Nil(R) \neq Z(R)$ such that $R(x)$ is an integrally closed ϕ-Mori ring even though $R/Nil(R)$ is not a Prüfer domain and neither R nor $R(x)$ is ϕ-Noetherian (Example 5.7).

Theorem 4.5. *Let R be an integrally closed ring for which $Nil(R) = Z(R) \neq \{0\}$. Then the following are equivalent.*

(1) *R is ϕ-Mori and NT is an ideal of $R(x)$.*
(2) *$R(x)$ is ϕ-Mori.*
(3) *$R(x)$ is ϕ-Noetherian.*
(4) *R is ϕ-Noetherian and NT is an ideal of $R(x)$.*
(5) *Each regular ideal of R is invertible.*
(6) *$R/Nil(R)$ is a Dedekind domain.*
(7) *R is a ϕ-Dedekind ring.*

Proof. Clearly, (3) implies (2), and (4) implies (1). By Theorem 4.3, we also have that if any one of (1) through (4) holds, then R is a Prüfer ring. The equivalence of (6) and (7) is by Theorem 2.5 of [3].

[(2)⇒(1)] Assume $R(x)$ is ϕ-Mori. Then NT is a divided prime of $R(x)$. Since $Nil(R) = Z(R)$, $NT = Z(R(x))$. Let I be a regular ideal of R. Then $I_vR(x) = (IR(x))_v$ [29, Lemma 5.1]. Moreover, $I_vR(x) \cap R = I_v$. Hence R is a Mori ring. As above, $R(x)/NT$ is naturally isomorphic to $(R/Nil(R))(x)$. Thus we also have that $R/Nil(R)$ is a Mori domain. Therefore R is a ϕ-Mori ring.

[(1)⇒(6)] Assume R is a ϕ-Mori ring and NT is an ideal of $R(x)$. Then $R/Nil(R)$ is both a Mori domain and a Prüfer domain. The only such integral domains are the Dedekind domains [31, Corollaire 2].

[(6)⇒(5)] Assume $R/Nil(R)$ is a Dedekind domain and let I be a regular ideal of R. Then R is a Prüfer ring by Theorem 4.3. Since R is integrally closed and $Nil(R) = Z(R)$, I (properly) contains $Nil(R)$. Moreover, as $I/Nil(R)$ is finitely generated, I must be finitely generated. Thus each regular ideal of R is invertible.

[(5)⇒(3)] Assume each regular ideal of R is invertible. Since R is a McCoy ring, both R and $R(x)$ are Prüfer rings [22, Corollary 18.11]. Thus each regular ideal of $R(x)$ is extended from an ideal of R [1, Theorem 3.3]. As an invertible ideal is always finitely generated, each regular ideal of $R(x)$ is invertible and finitely generated. Since $Nil(R) = Z(R)$ is a divided prime of R, $NT = Nil(R(x)) = Z(R(x)) = Nil(R)R(x)$ is a divided prime of $R(x)$. Moreover, each ideal that is not contained in $Nil(R(x))$ contains $Nil(R(x))$ and is regular. It follows that $R(x)$ is ϕ-Noetherian.

[(3)⇒(4)] Assume $R(x)$ is ϕ-Noetherian. Then NT is a divided prime of $R(x)$. Since $Nil(R) = Z(R)$, $NT = Z(R(x))$. Let I be a regular ideal of R. Then $IR(x)$ is a regular ideal of $R(x)$. Thus $IR(x)$ is finitely generated. As I generates $IR(x)$, I must be finitely generated as an ideal of R. Moreover, $R(x)/NT$ is naturally isomorphic to $(R/Nil(R))(x)$. So we also have that $R/Nil(R)$ is a Noetherian domain. Hence R is ϕ-Noetherian. □

Corollary 4.6. *Let R be a ring for which $Nil(R) = Z(R) \neq \{0\}$. If $R(x)$ is ϕ-Noetherian, then the integral closure of R is a ϕ-Dedekind ring.*

PROOF. Assume $R(x)$ is ϕ-Noetherian and let R' be the integral closure of R. Then $Nil(R(x)) = NT$ is a divided prime of both $R(x)$ and $R'(x)$. Moreover, since $R(x) \cap T(R) = R$, $Nil(R) = Nil(R')$ is a common divided prime of R and R'. By Corollary 4.4, R' is a Prüfer ring and $R'/Nil(R)$ is a Prüfer domain. Moreover, the proof that (3) implies (4) above is valid no matter whether R is integrally closed or not. Hence $R/Nil(R)$ is a Noetherian domain [10].

Since $Nil(R) = Z(R)$, we also have that $T(R)/Nil(R)$ is the quotient field of $R/Nil(R)$. From this we see that $R'/Nil(R)$ is the integral closure of $R/Nil(R)$. Thus $R/Nil(R)$ is a Noetherian domain whose integral closure is a Prüfer domain.

This means the integral closure is both a Prüfer domain and a Krull domain; i.e., it is a Dedekind domain. □

We can make a comparable statement for ϕ-Mori rings. However, the best conclusion is only that the integral closure of $R/Nil(R)$ is a Prüfer domain. It seems likely that in this case $R/Nil(R)$ (and therefore R) must be one-dimensional.

Example 4.7. Let D be a one-dimensional quasilocal Mori domain whose corresponding polynomial ring $D[x]$ is also Mori and whose integral closure is a nondiscrete one-dimensional valuation domain. Let $R = D(+)L$ be the ring formed by idealization of the quotient field of D. Then the following hold.

(a) $Nil(R) = Z(R)$ is a divided prime of R.
(b) R is a ϕ-Mori ring.
(c) $R(x)$ is a ϕ-Mori ring.
(d) The integral closure of $R/Nil(R)$ is a Püfer domain that is not a Dedekind domain.

PROOF. As in Example 2.11 above, we have that $Nil(R) = Z(R)$ is a divided prime of R and therefore R is a ϕ-Mori ring. Since $D[x]$ is a Mori domain, so is $D(x)$. As $D(x)$ is naturally isomorphic to $R(x)/Nil(R(x))$, we at least have that $R(x)/Nil(R(x))$ is a Mori domain. To complete the proof of statement (c) we must show that $Nil(R(x))$ is a divided prime of $R(x)$. Since the integral closure of D is a Prüfer domain, $L(x) = L[x]_{\mathcal{U}(D)}$ [26, Corollary 10]. It follows that $(0)(+)L(x)$ is the nilradical of $R(x)$. Thus $Nil(R(x))$ is a divided prime of $R(x)$. As the integral closure of D is a nondiscrete valuation domain, it is not Noetherian. □

Mori domains which satisfy all of the requirements in the previous example exist. In particular, W. Heinzer and D. Lantz describe a general scheme for constructing a one-dimensional quasilocal N-ring D whose integral closure is a nondiscrete rank one valuation domain [19, Example 2.2] (see also [27, Example 9]). A domain that is an N-ring is also a Mori domain [17, Corollary 2.8]. The domain D is a certain subring of a generalized power series ring of the form $K[[S]]$ where S consists of those positive rational numbers of the form $r + s/2^r$ where $1 \leq r$ and $s = 0, 1, 2, \ldots, 2^r - 1$ and K is an irredundant union of fields $K_0 \subset K_1 \subset K_2 \cdots$ where K_i is a finite algebraic extension of K_{i-1} for each $i \geq 1$. By starting with an uncountable field K_0 whose algebraic closure is not a finite extension, we will have a Mori domain that contains an uncountable field. For such a domain it is known that the corresponding polynomial ring is Mori [35, Theorem 3.15] (see also [13]).

Following the nomenclature above, a ring $R \in \mathcal{H}$ is said to be ϕ-integrally closed if $\phi(R)$ is integrally closed [3]. It is always the case that if $Nil(R)$ is divided prime of R, then $Nil(\phi(R)) = Z(\phi(R))$ is a divided prime of $\phi(R)$. Thus we may put together several of the results above to obtain the following corollary.

Corollary 4.8. *Let $R \in \mathcal{H}$. If R is ϕ-integrally closed and $\phi(R)$ is not an integral domain, then the following are equivalent.*

1 *R is ϕ-Dedekind.*
2 *$\phi(R)(x)$ is ϕ-Noetherian.*
3 *$\phi(R)(x)$ is ϕ-Mori.*
4 *$\phi(R)$ is ϕ-Mori and $Nil(T(\phi(R)[x]))$ is an ideal of $\phi(R)(x)$.*
5 *$\phi(R)/Nil(\phi(R))$ is a Dedekind domain.*
6 *$R/Nil(R)$ is a Dedekind domain.*

Theorem 4.9. *Let $R \in \mathcal{H}$. If $R(x)$ is a ϕ-ring and $\phi(R)$ is not an integral domain, then the integral closure of $\phi(R)$ is a Prüfer ring.*

PROOF. Assume $R(x)$ is a ϕ-ring and $\phi(R)$ is not an integral domain. Since $\phi(R)$ is a ϕ-ring with $Nil(\phi(R)) = Z(\phi(R)) \neq (0)$, it suffices to show that the nilradical of $\phi(R)(x)$ is divided (Corollary 4.4). For each $f(x) = \sum f_i x^i \in T(R)[x]$, let $\widehat{\phi}(f(x)) = \sum \phi(f_i)(x) \in R_{Nil(R)}[x]$. Then for each nonzero nilpotent polynomial $m(x) \in \phi(R)[x]$ and each nonnil polynomial $r(x) \in \phi(R)[x]$, there is a pair of polynomials $n(x) \in Nil(R)[x]$ and $s(x) \in R[x] \backslash Nil(R)[x]$ such that $m(x) = \widehat{\phi}(n(x))$ and $r(x) = \widehat{\phi}(s(x))$. Since $R(x)$ is a ϕ-ring, there is a nilpotent polynomial $k(x)$ and a polynomial $u(x)$ with unit content such that $u(x)n(x) = s(x)k(x)$. It follows that $v(x)m(x) = r(x)j(x)$ with $v(x) = \widehat{\phi}(u(x))$ having unit content in $\phi(R)$ and $j(x) = \widehat{\phi}(k(x))$ nilpotent. Since $v(x)$ is a unit in $\phi(R)(x)$, $m(x) \in r(x)\phi(R)(x)$. Thus $Nil(\phi(R)(x))$ is divided and, by Corollary 4.4, the integral closure of $\phi(R)$ is a Prüfer ring. □

5. EXAMPLES

All but one of the examples in this section are constructed using idealization of a particular type of divisible module, the construction of the other is only slightly different. Given an integral domain D and a nonempty set of nonzero prime ideals $\mathcal{P}$, let $B = \sum F/D_{P_\alpha}$ where F is the quotient field of D and the P_αs range over the set $\mathcal{P}$. It is easy to prove that B is a divisible D-module; i.e., $rB = B$ for each nonzero $r \in D$. From this it will follow that the idealized ring $R = D(+)B$ is in the set $\mathcal{H}$. We collect several useful facts about rings formed in this manner.

Theorem 5.1. *Let D be an integral domain with quotient field F and let $\mathcal{P} = \{P_\alpha\}$ be a nonempty set of nonzero prime ideals of D. For each $P_\alpha \in \mathcal{P}$, let $B_\alpha = F/D_{P_\alpha}$. Finally let $R = D(+)B$ where $B = \sum B_\alpha$. Then the following hold.*

(a) *$Z(R) = C(+)B$ where $C = \cup P_\alpha$.*
(b) *$T(R)$ can be identified with the ring $D_S(+)B$ where $S = D\backslash \cup P_\alpha$.*
(c) *For each nonzero $r \in D$ and each $b, c \in B$, there is an element $d \in B$ such that $(r,b)(0,d) = (0,c)$. Thus $(0)(+)B$ is a divided prime of R.*
(d) *If I is an ideal of R that is not contained in B, then it contains B and must be of the form $AR = A(+)B$ for some nonzero ideal of A of D.*
(e) *A finitely generated nonnil ideal $AR = A(+)B$ has a nonzero annihilator if and only if there is a prime $P_\alpha \in \mathcal{P}$ containing A such that $(D_{P_\alpha} : AD_{P_\alpha})$ properly contains D_{P_α}.*
(f) *If D is a Noetherian domain, then R is ϕ-Noetherian.*
(g) *If D is a Mori domain, then R is a Mori ring and a ϕ-Mori ring.*
(h) *$R(x)$ may be identified with the ring $D(x)(+)B(x)$ where $B(x) = B[x]_{\mathcal{U}(D)}$.*
(i) *If D is an integrally closed domain, then $R(x)$ is integrally closed if and only if $R(x)$ contains NT.*

In the proof below and throughout the examples that follow we identify B with $(0)(+)B$. For the proof below, we let D_α denote the ring D_{P_α}. Also for $b \in B$, we let b_α denote the component of b in B_α.

Theorem 25.1 of [22] gives many of the elementary properties of rings formed by idealization. For example, an element (u, c) is a unit of $A(+)C$ if and only if u is a unit of A, an element (r, a) is a zero divisor if and only if $r \in Z(A) \cup Z(C)$ where $Z(C) = \{r \in A \mid rb = 0$ for some nonzero $b \in C\}$, and $T(A(+)C)$ can be identified with the ring $A_S(+)C_S$ where $S = A\backslash(Z(A) \cup Z(C))$. The prime ideals of $A(+)C$ are the ideals of form $P(+)C$ where P is a prime ideal of A. These results originally appeared in [20, Section 4].

Proof. Since D is an integral domain and R is formed by the idealization of B over D, the zero divisors of R are the elements of the form (r, b) where $rc = 0$ for some nonzero $c \in B$. Moreover, the units of R are the elements of the form (u, b) where u is a unit of D. By the construction of B, $rc = 0$ for some $r \in D$ and $c \in B$, if and only if r is not a unit in some D_α. Hence $(r, b) \in Z(R)$ if and only if $r \in C = \cup P_\alpha$. It follows that $(s, c) \in R$ is not a zero divisor if and only if $s \in S = D\backslash \cup P_\alpha$. For such an element s, s is a unit in each D_α. Thus $s^{-1}B = B$ and therefore $T(R)$ can be identified with the ring $D_S(+)B$. Since D

is an integral domain and the prime ideals of R are the ideals of the form $P(+)B$ where P is a prime of D, B is a prime ideal of R.

Let $r \in D\backslash\{0\}$ and $b, c \in B$. We will work component-wise on those α for which c_α is not 0. Fix such an α and let $d_\alpha \in F\backslash D_\alpha$ be such that the image of d_α in B_α is c_α. Consider the element $r^{-1}d_\alpha$. Since $r \in D$ and d_α is not in D_α, $r^{-1}d_\alpha$ is not in D_α. The image of this element in B_α is $r^{-1}c_\alpha$. Let $r^{-1}c \in B$ be defined component-wise by $(r^{-1}c)_\alpha = r^{-1}c_\alpha$ when $c_\alpha \neq 0$ and $(r^{-1}c)_\beta = 0$ when $c_\beta = 0$. A simple check shows that $(r, b)(0, r^{-1}c) = (0, c)$. Hence B is a divided prime of R.

Since B is a divided prime of R, each ideal that contains a nonnilpotent element properly contains B. Thus the ideals of R are of two forms, $I = JR = J(+)B$ where J is a nonzero ideal of D and $(0)(+)E$ where E is a D-submodule of B. Let $AR = A(+)B$ be a finitely generated nonnil ideal of R. We first assume that AR has a nonzero annihilator in R. Since D is an integral domain, the annihilator of AR must be contained in B. If no P_α contains A, then $AD_\alpha = D_\alpha$. In which case, $Ab \neq (0)$ for each nonzero $b \in B$. As we have assumed AR does have nonzero annihilators, some P_α must contain A. Suppose P_α contains A and let $c_\alpha \in B_\alpha$ be such that $Ac_\alpha = (0)$. If $c_\alpha \neq 0$, then there is an element $d_\alpha \in F\backslash D_\alpha$ such that $d_\alpha A \subseteq D_\alpha$. Conversely if $f_\alpha \in F\backslash D_\alpha$ is such that $f_\alpha A \subseteq D_\alpha$, then A must be contained in P_α and the element of B whose only nonzero component is the image of $f_\alpha \in B_\alpha$ is a nonzero annihilator of AR. Thus AR has a nonzero annihilator if and only if $(D_\alpha : AD_\alpha) \neq D_\alpha$ for some α.

The regular divisorial ideals of R are all of the form $JR = J(+)B$ where J is a divisorial ideal of D that is not contained in $\cup P_\alpha$. Thus if D is a Mori domain, R will be a Mori ring. Since the nilradical of R is a divided prime, we also have that R is ϕ-Mori when D is a Mori domain (Theorem 2.5), and R is ϕ-Noetherian when D is Noetherian ([10, Theorem 2.2]).

For (h), first note that $R[x]$ is simply the idealization of $B[x]$ as a $D[x]$-module. Thus $T(R[x])$ can be identified with $D[x]_{\mathcal{S}}(+)B[x]_{\mathcal{S}}$ where $\mathcal{S}$ is the set of elements of $D[x]$ that are not zero divisors on $B[x]$.

This same identification lets us identify $R(x)$ with $D(x)(+)B(x)$ where $B(x) = B[x]_{\mathcal{U}(D)}$.

Obviously, if $R(x)$ is integrally closed, it must contain NT. Thus to finish the proof of (i) we may assume D is integrally closed and $R(x)$ contains NT. Since D is an integrally closed domain, both $D[x]$ and $D(x)$ are integrally closed. In particular, $D(x)$ is integrally closed in $D[x]_{\mathcal{S}}$. Thus $R(x)$ is integrally closed in $T(R[x])$ [22, Corollary 25.7]. □

In our first example of this section we use the technique outlined above to construct a ϕ-Noetherian ring R where $Nil(R) \neq (0)$ is properly contained in $Z(R)$, $R = T(R)$ and $R(x)$ is ϕ-Noetherian. Here, we start with a valuation domain V. In Example 5.4, we start with a UFD that is not a Prüfer domain and build a ring with similar properties.

Example 5.2. Let $R = V(+)B$ where $V = K[[y]]$ and $B = K((y))/V$. Then the following statements hold.

(a) $Z(R) = yK[[y]](+)B$.
(b) $R = T(R)$ and $Nil(R)$ is a divided prime of R with $R/Nil(R)$ isomorphic to V.
(c) R is ϕ-Noetherian.
(d) Let A be a finitely generated ideal of R. If $A \neq R$, then $Ann(A) \neq (0)$. Thus R is a McCoy ring.
(e) $Nil(R(x))$ is a divided prime of $R(x)$.
(f) $R(x)$ is ϕ-Noetherian.

PROOF. Since R is formed by idealization, each polynomial in $R[x]$ has a unique representation in the form $(g(x), b(x))$ with $g(x) \in V[x]$ and $b(x) \in B[x]$. Moreover, $R(x) = V(x)(+)B(x)$ where $B(x) = B[x]_{\mathcal{U}(V)}$.

Note that the first three statements are clear from the construction.

Let $A = ((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ be a finitely generated ideal of R with $A \neq R$. In any ring, a finitely generated ideal containing only nilpotents always has a nonzero annihilator. Hence we may assume some a_i is not 0. By Theorem 5.1, $A = IR$ where $I = (a_1, a_2, \ldots, a_n)V$ is a proper ideal of V. Thus there is a positive integer m such that $(a_1, \ldots, a_n) = y^m V$. Consider the product $(a_i, b_i)(0, 1/y)$. Since $a_i/y \in V$ for each i, $(a_i, b_i)(0, 1/y) = (0, 0)$. Thus A has a nonzero annihilator.

To prove (e), let $f(x) \in V[x]$ be a nonconstant polynomial whose content is not equal to V. Since V is a DVR, $f(x) = y^n u(x)$ for some positive integer n and some polynomial $u(x)$ with unit content in V. Thus for each nonzero nilpotent $(0, m)$ and each nilpotent $n(x) \in R(x)$, $(f(x), n(x))(0, m/y^n) = (0, m)/(u(x), 0) \in R(x)$. Therefore $Nil(R(x))$ is a divided prime of $R(x)$.

For (f), since R is a McCoy ring and $R = T(R)$, $R(x) = T(R[x])$ [22, Theorem 16.4]. So $R(x)$ is the only regular ideal of $R(x)$ and by (e), $Nil(R(x))$ is a divided prime of $R(x)$. Moreover, $R(x)/Nil(R(x))$ is isomorphic to $V(x)$ which is Noetherian. Therefore $R(x)$ is ϕ-Noetherian. □

As mentioned above, a Mori ring is said to be nontrivial if it is properly contained in its total quotient ring. Our next example is of a nontrivial Mori ring that is in the set $\mathcal{H}$ but is not a ϕ-Mori ring.

Example 5.3. Let E be a Dedekind domain with a maximal ideal M such that no power of M is principal (equivalently, M generates an infinite cyclic subgroup of the class group) and let $D = E + xF[x]$ where F is the quotient field of E. Let $\mathcal{P} = \{ND \mid N \in Max(E)\backslash\{M\}\}$ and let $R = D(+)B$. Then the following hold.

(a) If J is a regular ideal, then $J = I(+)B = IR$ for some ideal I that contains a polynomial in D whose constant term is a unit of E. Moreover, the ideal I is principal and factors uniquely as $P_1^{r_1}P_2^{r_2}\cdots P_n^{r_n}$ where the P_i are the height one maximal ideals of D that contain I.

(b) $R \neq T(R)$ since, for example, the element $(1+x, 0)$ is a regular element of R that is not a unit.

(c) R is a nontrivial Mori ring but R is not ϕ-Mori.

(d) MR is a maximal ϕ-divisorial ideal of R, but R_{MR} is not a Mori ring.

PROOF. Since E is a Dedekind domain, D is a Prüfer domain and $MD = M + xF[x]$ is an invertible maximal ideal of D (see [14]). By Theorem 5.1, the nilradical of R is divided and each regular ideal is of the form $IR = I(+)B$ where I is an ideal that is not contained in the union of the primes in $\mathcal{P}$. It is clear that if $g \in D$ is such that its constant term is not a unit of E, then $g \in ND$ for some $ND \in \mathcal{P}$. Hence if IR is regular it must contain a polynomial $g(x)$ whose constant term is a unit of E. Let P be a prime of D that contains I. Then $P = fF[x] \cap D$ where $f(x) \in E[x]$ has unit constant term. By checking locally, it is easy to see that $P = fD[x]$. Moreover, only finitely many primes of $F[x]$ contain $g(x)$ so the same is true for the primes of D. Thus I is contained in only finitely many primes of D, and each of these is a height one principal maximal ideal of D. Again, checking locally we see that I must factor uniquely as a finite product $P_1^{r_1}P_2^{r_2}\cdots P_m^{r_m}$ where the P_i are the primes of D that contain I.

The statement in (b) follows easily from (a). That R is a (nontrivial) Mori ring also follows easily from the characterization of its regular ideals. On the other hand R is not a ϕ-Mori ring since D is not a Mori domain.

In this case $\phi(R) = D$ and $\phi(MR) = M$, an invertible maximal ideal of D. Thus MR is ϕ-divisorial. Consider the localization R_{MR}. Each maximal ideal of E generates a maximal ideal of D that properly contains $xF[x]$. Since no power of M is principal, each nonzero nonunit of E is contained in at least one member of $\mathcal{P}$. Thus each element of B is annihilated by an element in $R\backslash MR$. Hence

$R_{MR} = D_M$, a two-dimensional valuation domain with divisorial maximal ideal MD_M. Such a domain is never Mori. □

Our next example is one of two promised earlier to show that the statements in Theorems 4.3 and 4.5 need not be equivalent when $Nil(R)$ is neither the zero ideal nor the entire set of zero divisors.

Example 5.4. Let $D = K[y, z]$ with K a field and let $\mathcal{P}$ denote the set of height one primes of D. For each $P_\alpha \in \mathcal{P}$, let $B_\alpha = K(y,z)/D_{P_\alpha}$. Let $R = D(+)B$ where $B = \sum B_\alpha$. Then the following hold.

(a) $Z(R) = \cup P_\alpha(+)B = (D\backslash K)(+)B$.
(b) $R = T(R)$ and it is a ϕ-Noetherian ring.
(c) $Nil(R) = (0)(+)B$ is a divided prime of R.
(d) $NT \subset R(x)$.
(e) NT is a divided prime of $R(x)$.
(f) $R(x)$ is integrally closed.
(g) $R(x)$ is a ϕ-Noetherian ring and a ϕ-Mori ring even though D is not a Dedekind domain.
(h) For each nonzero nilpotent element m, $Ann(m) = I(+)B$ where I is a finite product of positive powers of height one primes of D and $R/Ann(m)$ is a McCoy ring. If $I = P_\alpha^k$, then $Nil(R/Ann(m)) = Z(R/Ann(m))$. Otherwise, $Nil(R/Ann(m))$ is properly contained in $Z(R/Ann(m))$. The only time $Nil(R/Ann(m))$ is divided is when I is prime.

As in the proof of Theorem 5.1, we let D_α denote the ring D_{P_α} for each $P_\alpha \in \mathcal{P}$. Before starting the proof, some definitions are in order. An ideal I is said to be *semiregular* if it contains a finitely generated ideal J such that $Ann(J) = (0)$. A ring V is said to be a *discrete rank one valuation ring* if there is a surjective map $\psi : T(V) \to \mathbb{Z} \cup \{\infty\}$ (with $n < \infty = \infty + n = \infty + \infty$ for all n) such that $V = \{t \in T(V) \mid \psi(t) \geq 0\}$ and for all $s, t \in T(V)$, $\psi(st) = \psi(s) + \psi(t)$ and $\psi(s+t) \geq min\{\psi(s), \psi(t)\}$. Finally, a ring R is said to be a *Krull ring* if there is a family of discrete rank one valuation rings $\{V_\alpha \mid R \subset V_\alpha \subset T(R)\}$ (empty in the case $R = T(R)$) such that $R = \cap V_\alpha$ with each regular nonunit of R a unit in all but finitely many V_α. By allowing $R = T(R)$ to be a (trivial) Krull ring, we are following the lead of J. Huckaba in [22] rather than the original definition due to R. Kennedy where R is assumed to be properly contained in $T(R)$ [24].

PROOF. Since each nonunit of D is contained in at least one P_α, $f(+)b$ is a zero divisor for each $b \in B$ and $f \in D\backslash K$. Thus $R = T(R)$. Also $Nil(R)$ is a divided prime. The ideal $(y,z)(+)B$ contains only zero divisors but it has no nonzero annihilator. Thus R is not a McCoy ring.

Next we show that $R(x)$ contains NT, the nilradical of $T(R[x])$. To this end, let I be a finitely generated semiregular ideal of R and let b be a nonzero element of B. Since $Nil(R)$ is divided, $I = AR = A(+)B$ for some nonzero ideal A of D with A contained in no height one prime of D. For each α, let b_α denote the B_α component of b. Since $B = \sum B_\alpha$, there are only finitely many such α. Thus it suffices to prove the statement for those b with a single nonzero component, say b_β. Since P_β is a height one prime of a UFD, D_β is a discrete rank one valuation domain and $P_\beta = p_\beta D$ for some $p_\beta \in D$. Thus we may further assume b_β has the form $1/p_\beta^m$ for some positive integer m. In this case, the annihilator of b is the ideal $p_\beta^m D(+)B$. Thus $R/Ann(b)$ is naturally isomorphic to $D/p_\beta^m D$, a one-dimensional Noetherian ring with prime nilradical $P_\beta/p_\beta^m D$. Now the integral closure of each Noetherian ring is a Krull ring [22, Theorem 10.1]. Moreover, as with one-dimensional Krull domains (i.e., Dedekind domains), each regular finitely generated ideal of a one-dimensional Krull ring is invertible. To prove this simply use finite character and the fact that the regular localization at a height one regular prime yields a discrete rank one valuation ring [22, Theorem 8.10]. Thus $A/p_\beta^m D$ generates an invertible ideal in the integral closure of D/p_β^m. Since this happens for each finitely generated semiregular ideal of R and each nonzero nilpotent element, $NT \subset R(x)$ [25, Theorem 8] (see also [25, Theorem 6]).

Next we show that NT is actually a divided prime of $R(x)$. There is nothing to prove for the regular elements of $R(x)$ since each divides each member of NT. Thus we may assume $(r(x), b(x))$ is such that the content of $r(x)$ is a nonzero ideal of D that is contained in some P_α. Since $D = K[y,z]$, we may write $r(x) = p_{\alpha_1}^{m_1} p_{\alpha_2}^{m_2} \cdots p_{\alpha_n}^{m_n} s(x)$ where each $m_i \geq 1$ and the content of $s(x)$ is contained in no height one prime of D. Since such an $s(x)$ divides each member of NT, we may assume $s(x) = 1$. To complete the proof we simply use the fact that each member of $Nil(R)$ is divisible by $p_{\alpha_1}^{m_1} p_{\alpha_2}^{m_2} \cdots p_{\alpha_n}^{m_n}$. Therefore NT is a divided prime of $R(x)$.

Since $D(x)$ is an integrally closed Noetherian domain and NT is a divided prime of $R(x)$, $R(x)$ is integrally closed ([22, Corollary 25.7], or Theorem 5.1 above), each regular ideal of $R(x)$ is finitely generated and $R(x)$ is both ϕ-Noetherian and ϕ-Mori.

Let m be a nonzero nilpotent element and let $m_{\alpha_1}, m_{\alpha_2}, \ldots, m_{\alpha_n}$ be the nonzero components of m. For each α_i there is a positive integer k_i and a unit $v_i \in D_{\alpha_i}$ such that $m_{\alpha_i} = (v_i/p_{\alpha_i}^{k_i}) + D_{\alpha_i}$ where, as above, each p_{α_i} generates the prime ideal P_{α_i}. It follows that $Ann(m) = I(+)B$ where $I = P_{\alpha_1}^{k_1} P_{\alpha_1}^{k_2} \cdots P_{\alpha_n}^{k_n}$. Thus $R/Ann(m)$ is isomorphic to the Noetherian ring D/I. It is well-known that in a Noetherian ring, all ideals containing only zero divisors have nonzero annihilators. Thus $R/Ann(m)$ is a McCoy ring. For the other parts of (h), we need only consider the Noetherian ring D/I.

The nilradical of D/I is simply $\sqrt{I}/I$ where $\sqrt{I} = p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_n} D$ while $Z(D/I)$ is the union $(P_{\alpha_1}/I) \cup (P_{\alpha_2}/I) \cup \cdots \cup (P_{\alpha_n}/I)$. It follows that $Nil(D/I) = Z(D/I)$ if and only if $n = 1$. Similarly, $Nil(D/I)$ is prime if and only if $n = 1$. Moreover, $Nil(D/I)$ is a divided prime if and only if $I = P_{\alpha_1}$ is prime. □

Next we give an example where R is an integrally closed ϕ-Noetherian ring, but $R(x)$ is not.

Example 5.5. Let $D = K[w, y, z]$ with w, y and z indeterminates over a field K and let $\mathcal{P}$ denote the set of height one primes of D. For each $P_\alpha \in \mathcal{P}$, let B_α denote $K(w, y, z)/D_{P_\alpha}$. Let $R = D(+)B$ where $B = \sum B_\alpha$. Then the following hold.

(a) $Z(R) = \cup P_\alpha(+)B = (D \backslash K)(+)B$.
(b) $R = T(R)$.
(c) $Nil(R) = (0)(+)B$ is a divided prime of R.
(d) $R(x)$ is not a Mori ring nor does it contain NT. Thus $R(x)$ is not ϕ-Noetherian.

PROOF. The proof for each of the first three statements follows the same line of reasoning as that used to prove the corresponding statements in the previous example. Consider the height one prime $P_\beta = wD$ and let $b \in B$ be such that $b_\beta = 1/w$ and $b_\alpha = 0$ for all other α. To show that $R(x)$ does not contain NT it suffices to show that $(0, b)/(yx + z, 0)$ is not in $R(x)$. Now the annihilator of $(0, b)$ is the ideal $wD(+)B$. Thus $R/Ann(0, b)$ is naturally isomorphic to $D/wD = K[y, z]$, a two-dimensional Krull domain. Obviously, $(y, z)K[y, z]$ is not invertible. Therefore, $(0, b)/(yx + z, 0)$ is not in NT [25, Theorem 8].

For each positive integer n, let $I_n = ((yx + z, 0), (0, b), (0, b^2), \ldots, (0, b^n))$ and consider the ascending chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$. Since no height one prime of D contains both y and z but $(y, z)D \neq D$, $(yx + z, 0)$ is a regular nonunit of $R(x)$. Obviously, any element of the form $(f(x)/yx + z, 0)$ will multiply $(yx + z, 0)$ into $R(x)$. Next, note that for $n \leq k$, $w^k b^n = 0$ but $w^{n-1} b^n = b$. Thus for $j \leq n-1$, $(w^{n-1}/yx + z, 0)(0, b^j) = (0, 0) \in R(x)$ but $(w^{n-1}/yx + z, 0)(0, b^n) = (0, b)/(yx + z, 0)$ is not in $R(x)$. Hence $(w^{n-1}/yx + z, 0)$ is in I_{n-1}^{-1} but not in I_n^{-1}.

It follows that $(I_1)_v \subset (I_2)_v \subset (I_3)_v \subset \cdots$ is a strictly increasing chain of regular divisorial ideals of $R(x)$. Thus $R(x)$ is not a Mori ring. □

The construction in our next example is a variation on the general form we have been dealing with. The difference is that instead of taking $B = F/D_Q$ for some prime Q, we take $B = F/QD_Q$. Except for the characterization of the ideals with nonzero annihilators, the basic properties established in Theorem 5.1 hold for a ring of the form $R = D(+)(F/QD_Q)$. In this case, QR will have a nonzero annihilator no matter what $(D_Q : QD_Q)$ is. As in the previous examples, we start with a polynomial ring over a field. In this case, the domain D is an intersection of two incomparable localizations of $K[y, z]$.

Example 5.6. Let $D = K[y,z]_{(y+1)} \cap K[y,z]_{(y,z)}$ and let $P = (y+1)K[y,z]_{(y+1)} \cap D$ and $M = (y,z)K[y,z]_{(y,z)} \cap D$. The following statements hold for the ring $R = D(+)B$ where $B = K(y,z)/MD_M$.

(a) $Z(R) = M(+)B$ and $T(R) = D_M(+)B$.
(b) B is a divisible D-module so R is ϕ-Noetherian.
(c) $MR = M(+)B$ has a nonzero annihilator, so R is a McCoy ring.
(d) Each regular ideal of R is of the form $P^nR = P^n(+)B$, a principal ideal. Thus R is a Prüfer ring.
(e) $R(x)$ contains the nilradical of $T(R[x])$, but $Nil(R(x))$ is not divided.

PROOF. For each $d \in K(y,z)$, let $\overline{d}$ denote its image in B. Then for each $d \in D \backslash M$, $M\overline{d} = 0$ in B. Hence $M(+)B$ has a nonzero annihilator. On the other hand, $y+1$ annihilates no nonzero element of B. Since P and M are the only maximal ideals of D and P has height one, $Z(R) = M(+)B$. Also, $P(+)B$ is regular. Let $q \in K(y,z)\backslash MD_M$. Then $\overline{(q/r)} \neq 0$ in B for each nonzero $r \in D$. Since $r\overline{(q/r)} = \overline{q}$, B is a divisible D-module. Thus $Nil(R)$ is a divided prime and R is a ϕ-ring by Theorem 5.1. Also $P(+)B = PR$ is a principal ideal since $P = (y+1)D$. Since D is Noetherian, R is ϕ-Noetherian.

Let A be a proper nonnil ideal of R. Since $Nil(R)$ is divided, $A = JR = J(+B)$ for some nonzero ideal J of D. If $J \subset M$, then A has a nonzero annihilator. On the other hand if J is not contained in M, then it must be P-primary and A is regular. Since P and M are the only maximal ideals of D and P is height one and principal, each P-primary ideal is principal and a power of P. It follows that each regular ideal of R is invertible. Thus R is both a McCoy ring and a Prüfer ring. This combination is enough to guarantee that $R(x)$ is Prüfer so $R(x)$ contains the nilradical of $T(R[x])$ (see [22, Corollary 18.11]).

It remains to show that $Nil(R(x))$ is not divided. For this we show that $\overline{1}$ cannot be divided by $yx+z$ in $R(x)$. Assume otherwise. Then by Theorem 5.1 there must be a polynomial $u(x) \in D[x]$ with unit content in D and a polynomial $f(x) \in K(y,z)[x]$ such that $u(x)\overline{1} = (yx+z)\overline{f}(x)$. Write $u(x) = u_0 + u_1 + \cdots + u_m x^m$ and $\overline{f} = \overline{f}_0 + \overline{f}_1 x + \cdots + \overline{f}_n x^n$ with the leading coefficient $\overline{f}_n$ not zero; i.e., $f_n \notin MD_M$. Without loss of generality we may assume some coefficient of $u(x)$ is 1. Thus for some i, $\overline{u}_i = \overline{1} = y\overline{f}_{i-1} + z\overline{f}_i$. Since each u_j is in D, $z\overline{f}_0, y\overline{f}_n \in D_M/MD_M$ and inductively we have $z^{k+1}\overline{f}_k, y^{k+1}\overline{f}_{n-k} \in D_M/MD_M$. It follows that $f_j \in (D_M : M^{n+1}D_M)$ for each j. But since D_M is a two-dimensional integrally closed Noetherian domain, $(D_M : M^{n+1}D_M) = D_M$. This implies $(yx+z)\overline{f}(x) = 0$ a contradiction. Therefore $Nil(R(x))$ is not divided. □

For our last example we show how to construct an integrally closed ϕ-Mori ring R with $R(x)$ ϕ-Mori and $R/Nil(R)$ an integrally closed Mori domain that is neither Noetherian nor Prüfer.

Example 5.7. Let $\mathcal{X} = \{x_\alpha\}$ be a set of algebraically independent indeterminates over a field K and for some fixed x_β, let $D = K[\mathcal{X}]_{M'} \cap K[\mathcal{X}]_{(x_\beta+1)}$ where M' is the maximal ideal of $K[\mathcal{X}]$ generated by the set $\mathcal{X}$. Finally let $R = D(+)B$ where $B = K(\mathcal{X})/D_P$ for $P = (x_\beta+1)D$. Then the following hold.

- (a) D is a Krull domain with two maximal ideals, $M = M'D$ and $P = K[\mathcal{X}]_{M'} \cap (x_\beta+1)K[\mathcal{X}]_{(x_\beta+1)}$.
- (b) D is Noetherian if and only if $\mathcal{X}$ is finite. Moreover, D is Dedekind if and only if $\mathcal{X}$ is a singleton set.
- (c) $Z(R) = P(+)B$ and $Nil(R) = (0)(+)B$ is a divided prime of R.
- (d) $R \neq T(R) = D_P(+)B$.
- (e) $NT \subseteq R(x)$ is a divided prime of $R(x)$.
- (f) $R(x)$ is an integrally closed ϕ-Mori ring, but it is a ϕ-Noetherian ring if and only if $\mathcal{X}$ is a finite set.

PROOF. Since M is a maximal ideal, each element of $P\backslash M$ is comaximal with M. Also, for each element $f \in M\backslash P$ and each positive integer m, $f + (x_\beta+1)^m$ is in neither M nor P. In particular f and P are comaximal. It follows that D is a Krull domain with exactly two maximal ideals with P a principal maximal ideal (necessarily, of height one). As no element of $\mathcal{X}$ is in P, the elements of the form $(x_\alpha, 0) \in M\backslash P$ are regular elements of R that are not units of R. Thus $R \neq T(R)$.

It is easy to see that $Z(R) = P(+)B$ and that $Nil(R) = (0)(+)B$ is a divided prime of R. Note that $P(+)B$ annihilates the nonzero nilpotent $(0, 1/(x_\beta + 1)$. Thus there are no semiregular ideals that are not regular.

Let $(f(x), c(x))$ be a nonnilpotent element of $R[x]$ and let $(0, b)$ be a nonzero nilpotent element of R. By the construction of D, there is a nonnegative integer m such that $f(x) = (x_\beta + 1)^m g(x)$ where $g(x) \in D[x] \backslash P[x]$. Let $k = deg(g(x))$. Then for each positive integer q, $g(x) + (x_\beta + 1)^q x^{k+1}$ is a polynomial with unit content in D. Since D_P is a discrete rank one valuation domain and b is not zero, there is a nonpositive integer n such that $b = u(x_\beta + 1)^n$ with u a unit of D_P. Set $d = u(x_\beta + 1)^{n-m} \in B$. Then $f(x)d = g(x)b = (g(x) + (x_\beta + 1)^{-n+1} x^{k+1})b$ since $(x_\beta + 1)^{n+1} b = 0$. Since $v(x) = g(x) + (x_\beta + 1)^{-n+1} x^{k+1}$ has unit content in D, $(0, d)(v(x), 0)$ is in $R(x)$ and we have $(0, b) = (f(x), c(x))(0, d)/(v(x), 0)$. It follows that NT is a divided prime of $R(x)$. Since D is a Krull domain, both it and $D(x)$ are integrally closed Mori domains. Thus by Theorem 2.5 and 5.1, $R(x)$ is an integrally closed ϕ-Mori ring.

Since D is Noetherian if and only if $\mathcal{X}$ is finite, we have that R and $R(x)$ are ϕ-Noetherian if and only if $\mathcal{X}$ is finite ([10, Theorem 2.2], or Theorem 5.1 above). □

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